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ON AN INTEGRABLE CASE OF PERTURBED KEPLERIAN MOTION*

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A general solution of a differential vector equation of perturbed Keplerian motion is derived for the case when the position vector and perturbing acceleration vector are collinear. A variable change is employed, in which the new independent variable is expressed in terms of the initial values of the phase variables and time, using the elliptical Jacobi function. The two-point boundary value problem for the initial equation is reduced to the Cauchy problem. A parametric representation is obtained for the regularized trajectory of motion of a material point under the action of a central force.

Let us consider a differential vector equation of perturbed Keplerian motion

$$\mathbf{r}^{\cdot\cdot} = -\mu\mathbf{r}r^{-3} + w\mathbf{r} \tag{1}$$

in which $\mathbf{r}, \mathbf{r}^{\cdot\cdot}$ are the vectors of position and acceleration of a material point, μ is the gravitational constant of the centre of attraction and w is a constant.

The differential Eq.(1) determines the intermediate orbits of a geocentric satellite four-body problem /1/, and of the known geocentric planetary problem of n bodies /2/.

A general integral of the equation of the type (1) appears in a number of papers (e.g. in /3/), but is not solved for the required coordinates of the vector $\mathbf{r}(x, y, z)$.

We shall assume that the following initial conditions are specified in the initial coordinate system for the instant $t = t_0$:

$$\mathbf{r}(t_0) = (x_0, y_0, z_0), \quad \mathbf{r}^{\cdot}(t_0) = (x_0^{\cdot}, y_0^{\cdot}, z_0^{\cdot}).$$

Let us bring into our discussion the constant vector of angular momentum and the oscillating Laplace vector

$$\mathbf{h} = [\mathbf{r}, \mathbf{r}^{\cdot}] = (h_1, h_2, h_3), \quad \mathbf{l} = [r^{\cdot}, h] - \mu\mathbf{r}r^{-1}$$

The differential equation for $\mathbf{l}$ now takes the form

$$\mathbf{l}^{\cdot} = -\mu^{-1}wr[\mathbf{l}, \mathbf{h}]$$

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To solve this equation in finite form, is its best to introduce a new parameter, namely the regular time ξ defined by the differential relation

$$d\xi = -\mu^{-1} wr dt \tag{2}$$

Then, provided that $w \neq 0, h \neq 0$, the equivalent system of linear differential equations with constant coefficients $d\mathbf{l}/d\xi = [\mathbf{l}, \mathbf{h}]$ will integrate according to the rules of operational calculus /4/.

As a result we obtain

$$l_1(\xi) = l_{10}\cos h\xi + (l_{20}h_3 - l_{30}h_2)\,h^{-1}\sin h\xi + (l_{10}h_1^2 + l_{20}h_1h_2 + l_{30}h_1h_3)\,h^{-2}(1 - \cos h\xi) \quad (1\ 2\ 3) \tag{3}$$

where the quantities $l_1, l_2, l_3, l_{10}, l_{20}, l_{30}$ are, respectively, the running and initial values of the components of the Laplace vector, and the relations not written out can be obtained by cyclic permutation of the indices 1, 2, 3.

In what follows we shall make use of the energy integral

$$\frac{l^2}{2h^2} - \frac{\mu^2}{2h^2} - \frac{w}{2}r^2 = H \tag{4}$$

From (4) we obtain

$$r = \left[\frac{2}{w}\left(\frac{l^2}{2h^2} - \frac{\mu^2}{2h^2} - H\right)\right]^{1/2} \tag{5}$$

Let us now pass to the process of determining the values of the parameter ξ. By virtue of formula (2) we have the integral

$$t = t_0 - \frac{\mu}{w}\int_0^{\xi}\frac{d\xi}{r} \tag{6}$$

Substituting (3) and (4) into (5) we obtain

$$r = (A\cos^2 h\xi + B\sin h\xi\cos h\xi + C\cos h\xi + D\sin h\xi + G)^{1/2}$$

where A, B, C, D, G are constant coefficients. After carrying out the trigonometric substitution $\xi = 2h^{-1}\,\mathrm{arctg}\,\sigma$, the integral in ($\sigma$) becomes an elliptic type integral in σ:

$$\int_0^{\xi}\frac{d\xi}{r} = \frac{2}{h}\int_0^{\sigma}\frac{d\sigma}{\sqrt{\Phi}}, \quad \Phi(\sigma) = r^2(1+\sigma^2)^2 \tag{7}$$

where ($\Phi(\sigma)$ is a fourth-degree polynomial.

Taking (7) into account, we write relation (6) in the form

$$\int_0^{\sigma}\frac{d\sigma}{\sqrt{\Phi}} = \frac{wh}{2\mu}(t_0 - t)$$

From the theory of elliptic integrals it follows that by carrying out the substitution $\sigma = (p + q\eta)/(1+\eta)$, we arrive at the standard expansion of the integrand

$$(q-p)\int_{\beta}^{\eta}\frac{d\eta}{[\varkappa(\eta^2+a^2)(\eta^2+b^2)]^{1/2}} = \frac{wh}{2\mu}(t_0 - t) \tag{8}$$

and from formula (8) we obtain /5/

$$\eta = b\,\mathrm{tn}\left(\frac{awh\sqrt{\varkappa}}{2(q-p)\mu}(t_0 - t) + F(\alpha, k)\right)$$

where $F(\alpha, k)$ is an incomplete elliptic integral of the first kind.

We note that the quantities α, β and the modulus of the elliptic function $\mathrm{tn}(\cdot)$ are as follows:

$$\alpha = \mathrm{arctg}\left(-\frac{p}{qb}\right), \quad \beta = -\frac{p}{q}, \quad k = \left(\frac{a^2 - b^2}{b^2}\right)^{1/2}$$

Next, using the relations of /3/ and formulas (2) and (5), we find the true anomaly θ from the values

$$\cos\theta = \frac{h^2}{lr} - \frac{\mu}{l}, \quad \sin\theta = -\frac{hwr}{\mu l}\frac{dr}{d\xi}$$

The existing formulas /3/ allow us to determine the longitude of the ascending node, the argument of the pericentre, real longitude, and then, using the relations obtained, the coordinates and velocity components of the material point.

Let us formulate the boundary value problem of determining the solution of Eq.(1) in terms of the given values of $\mathbf{r}(t_0)=\mathbf{r}_0$, $\mathbf{r}(t_1)=\mathbf{r}_1$ and $\Delta t=t_1-t_0$. We shall assume that the vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ are not collinear. It must be noted that an analogous boundary value problem was solved in detail for Keplerian motion in /2/. Let us determine $x_0^{\cdot}, y_0^{\cdot}, z_0^{\cdot}$. Using the relation /3/ $r^{\cdot}=Mr^{-1}$, $M=[wr^4+2Hr^2+2\mu r-h^2]^{1/2}$ and the integral of kinematic momentum $r^2\theta^{\cdot}=h$, we obtain the following relations:

$$\Delta t=\int_{r_0}^{r_1}\frac{r\,dr}{M},\quad \Delta\theta=h\int_{r_0}^{r_1}\frac{dr}{rM} \tag{9}$$

We note that the angle between the vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ is equal to $\Delta\varphi=\Delta\theta+2\pi m$ where m is an integer. Using the normal elliptic integrals I_1, I_{-1} /6/, we shall write relations (9) in the form

$$\Delta t=I_1(r,h,H)\big|_{r_0}^{r_1},\ \Delta\varphi=hI_{-1}(r,h,H)\big|_{r_0}^{r_1}+2\pi m$$

The relations obtained form a system of non-linear equations in h and H. Let us determine the coefficient of proportionality $c=h^{-1}|[\mathbf{r}_0,\mathbf{r}_1]|$, and then, using the condition of codirectionality of $\mathbf{h}$ and $[\mathbf{r}_0,\mathbf{r}_1]$, find $\mathbf{h}=c^{-1}[\mathbf{r}_0,\mathbf{r}_1]$. Next, using the quantities $x_0, y_0, z_0, h_1, h_2, h_3, r_0^{\cdot}$ we obtain, in accordance with /3/, $x_0^{\cdot}, y_0^{\cdot}, z_0^{\cdot}$.

In conclusion we shall consider the motion of a material point whose acceleration has a central component only

$$f(r)=-\mu r^{-2}+g(r)$$

Let us write the energy integral in the form

$$x^{\cdot 2}+y^{\cdot 2}+z^{\cdot 2}=2\mu r^{-1}-2V(r)+2H,\quad V(r)=-\int g(r)\,dr$$

Using the KS-transformation /7/ $(x,y,z,0)^T=L(\mathbf{u})\,\mathbf{u}$, we obtain the regularized equation

$$\frac{d^2\mathbf{u}}{ds^2}=\frac{H}{2}\mathbf{u}-\frac{1}{4}\frac{\partial}{\partial\mathbf{u}}(u^2V(u^2)),\quad s=\int_{t_0}^{t}r^{-1}(t)\,dt \tag{10}$$

(the Zundman variable s is used as the argument).

As a result of the substitution

$$d\tau=[1-H^{-1}(V(u^2)-u^2g(u^2))]^{1/2}\,|\,ds\,| \tag{11}$$

the solution of Eq.(10) can be written in the regularized Keplerian form /7/

$$\mathbf{u}(\tau)=c_0(-{}^1/_2H\tau^2)\,\mathbf{u}_0+\tau c_1(-{}^1/_2H\tau^2)\,(\mathbf{u}_\tau')_0 \tag{12}$$

When $dt>0$, we obtain from the second equation of (10) and (11)

$$t=t_0+\int_{\tau_0}^{\tau}\frac{u^2\,d\tau}{[1-H^{-1}(V(u^2)-u^2g(u^2))]^{1/2}}$$

Relations (11) and (12) determine the exact parametric solution of the Cauchy problem of Eq.(10).

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